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Exact *m*-covers of Groups by Cosets

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Let G be a group covered by its left cosets $a_1G_1, \dots, a_kG_k$ exactly m times. It is known that $[G : \bigcap_{i=1}^k G_i] \leqslant k!$. When all the G_i are subnormal in G and $\bigcap_{i=1}^k G_i = H$, we are able to determine the least value of k in terms of m, G, H. For any $i = 1, \dots, k$, providing $G/(G_i)_G$ is solvable we show that $k \geqslant m + f([G : G_i])$ and hence $[G : G_i] \leqslant 2^{k-m}$, where $f(n) = \sum_{s=1}^r \alpha_s(p_s - 1)$ if $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ is the standard factorization of n. These extend some previous results on disjoint covers.

1. INTRODUCTION

Let G be a (multiplicative) group (whose identity element is denoted by e) and H a subgroup of G. For any $a \in G$ we call $aH = \{ax : x \in H\}$ a left coset of H in G, by G/H we mean *the set of all left cosets of H in G*. As usual, the index of H in G is $[G : H] = |G/H|$, and G/H is called the quotient group (or the factor group) of G by H if H is normal in G. We use H_G and H^G to denote the core (i.e., normal interior) and the normal closure of H in G, respectively.

Let $a_1G_1, \dots, a_kG_k$ be finitely many left cosets in a group G. For the system

$$\mathcal{A} = \{a_iG_i\}_{i=1}^k, \tag{1.1}$$

we call

$$w_{\mathcal{A}}(x) = |\{1 \leqslant i \leqslant k : x \in a_iG_i\}| \tag{1.2}$$

the *covering multiplicity* of $x \in G$, and put $m(\mathcal{A}) = \inf_{x \in G} w_{\mathcal{A}}(x)$.

Let $m \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$. (1.1) is said to be an *exact m-cover* of G if $w_{\mathcal{A}}(x) = m$ for all $x \in G$. Exact 1-covers are just partitions into left cosets, they are also called *disjoint covers*. As mentioned in all textbooks on group theory, for any subgroup H of G with finite index, all the $[G : H]$ left cosets of H in G form a disjoint cover of G. Clearly the only exact m-cover of G by subgroups consists of m copies of G. It is known that even for the additive group $\mathbb{Z}$ of integers there exists an exact m-cover no subsystem of which forms an exact n-cover with $0 < n < m$ (cf. [4, 24]). Exact m-covers are natural extension of disjoint covers, they should have regular properties because any such a cover covers all the elements the same times. Covers of $\mathbb{Z}$ by residue classes were initiated by Erdös [2] (see Section 1 of Sun [20] for problems and results in the area), exact m-covers of $\mathbb{Z}$ were first investigated by Porubský [14], exact m-covers of an arbitrary group have been studied only in the case $m = 1$.

Exact 1-covers are very special in the sense that members of such a cover are pairwise disjoint. There has been much research on such covers, see [12, 15, 27] for results concerning disjoint covers of $\mathbb{Z}$. Often there are no obvious ways to generalize results on disjoint covers to exact m-covers, this is why few properties of exact m-covers are known. By an easy counting argument, if a system

$$A = \{a_i + n_i\mathbb{Z}\}_{i=1}^k \quad (a_1, \dots, a_k \in \mathbb{Z};\ n_1, \dots, n_k \in \mathbb{Z}^+) \tag{1.3}$$

of residue classes forms an exact m-cover of $\mathbb{Z}$ then $\sum_{i=1}^k 1/n_i = m$. In [19–21] the author revealed some further connections between exact m-covers of $\mathbb{Z}$ and unit fractions. In this paper we shall prove some inequalities for exact m-covers of groups.

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The Mycielski function $f : \mathbb{Z}^+ \to 0, 1, 2 \dots$ is determined as follows:

$$f(p) = p - 1 \text{ for any prime } p \text{ and } f(mn) = f(m) + f(n) \text{ for all } m, n \in \mathbb{Z}^+.$$

Evidently

$$f(p_1^{\alpha_1} \cdots p_r^{\alpha_r}) = \sum_{s=1}^{r} \alpha_s (p_s - 1) \tag{1.4}$$

where $p_1, \dots, p_r$ are distinct primes and $\alpha_1, \dots, \alpha_r$ are non-negative integers. In 1966 Mycielski and Sierpiński [9] conjectured that if (1.1) forms a disjoint cover of an abelian group G and all the $[G : G_i]$ are finite then

$$k \geqslant 1 + \max_{1 \leqslant i \leqslant k} f([G : G_i]). \tag{1.5}$$

This was confirmed by Znám [25] for $G = \mathbb{Z}$, and verified by Hejńy and Znám [5] in a special case. In 1968 Znám [26] posed a conjecture that if (1.3) forms a disjoint cover of $\mathbb{Z}$ then $k \geqslant 1 + f(N)$ where N is the least common multiple of $n_1, \dots, n_k$. Later in 1974 his student Korec ([6]) obtained the following stronger result: Let G be a group and (1.1) its disjoint cover with all the G_i normal in G, then each G_i has finite index in G and

$$k \geqslant 1 + f\left(\left[G : \bigcap_{i=1}^{k} G_i\right]\right). \tag{1.6}$$

In 1988 Berger *et al.* [1] proved that (1.5) holds if (1.1) forms a disjoint cover of a finite solvable group G.

It seems that Mycielski, Znám and Korec had not known the following basic result established by Neumann [10, 11] in 1954: if (1.1) forms a cover of a group G then G is the union of those $a_i G_i$ with $[G : G_i] < \infty$. In 1987 Tomkinson [23] strengthened the Neumann result by showing that if (1.1) forms a cover of a group G but none of its proper subsystems does then

$$\left[G : \bigcap_{i=1}^{k} G_i\right] \leqslant k! \tag{1.7}$$

where the upper bound $k!$ is the best possible. By Corollary 1 of Sun [18], (1.7) holds if $m(\mathcal{A}') < m(\mathcal{A}) = m$ for any proper subsystem $\mathcal{A}'$ of $\mathcal{A}$. In particular, when (1.1) forms an exact m-cover of group G, all the G_i and hence the intersection $\bigcap_{i=1}^{k} G_i$ are of finite index in G.

Let G be a group and H a subnormal subgroup of G with finite index. Let

$$H_0 = H \subset H_1 \subset \cdots \subset H_n = G$$

be a composition series from H to G. If the length n is zero (i.e., $H = G$), then we set $d(G, H) = 0$, otherwise we put

$$d(G, H) = \sum_{i=0}^{n-1} ([H_{i+1} : H_i] - 1). \tag{1.8}$$

By the Jordan–Hölder theorem, $d(G, H)$ does not depend on the choice of the composition series from H to G. Theorem 6 of [18] indicates that

$$[G : H] - 1 \geqslant d(G, H) \geqslant f([G : H]) \geqslant \log_2 [G : H]. \tag{1.9}$$

In contrast with Tomkinson's result, inequality (1.6) implies that

$$\left[G : \bigcap_{i=1}^{k} G_i\right] \leqslant 2^{k-1}. \tag{1.10}$$

In 1990 the author [18] obtained the following improvement to Korec's result: if (1.1) forms a disjoint cover of a group G with all the G_i subnormal in G, then

$$k \geqslant 1 + d\left(G, \bigcap_{i=1}^{k} G_i\right). \tag{1.11}$$

Let us look at two examples.

EXAMPLE 1.1. Let G be a group and H a subgroup with $k = [G : H] < \infty$. Suppose that $\{Ha_1, \ldots, Ha_k\}$ is a right coset decomposition of G. Set $G_i = a_i^{-1}Ha_i$ for $i = 1, \ldots, k$. The author [18] observed that (1.1) forms a disjoint cover of G with $\bigcap_{i=1}^{k} G_i = H_G$. When $G = S_k$ is the symmetric group on $\{1, \ldots, k\}$ and H is the stabilizer of 1,

$$\{H,\ H(12),\ H(13), \ldots, H(1k)\} = \{G_1,\ (12)G_2, \ldots, (1k)G_k\}$$

forms a partition of G where G_i is the stabilizer of i for each $i = 1, \ldots, k$, Tomkinson [23] noticed that in this case $\bigcap_{i=1}^{k} G_i = H_G = \{e\}$ has index $k!$ in G; we add here that if $k \geqslant 3$ then all the subgroups $G_1, \ldots, G_k$ are distinct, $[G : \bigcap_{i=1}^{k} G_i] = k!(> 2^{k-1})$ has some prime divisors (e.g., prime factors of $k-1$) not dividing $k = [G : G_i]$, and $f([G : \bigcap_{i=1}^{k} G_i]) = \sum_{i=2}^{k} f(i) \geqslant f(3) + k - 2 = k$.

EXAMPLE 1.2. Let G be a group and $H = H_0 \subset H_1 \subset \cdots \subset H_n = G$ be a chain of subgroups with finite index. Let $a \in G$ and $H_{i+1} \setminus H_i = \bigcup_{j=1}^{[H_{i+1}:H_i]-1} b_j^{(i)} H_i$ for $i = 0, 1, \ldots, n-1$. In [18] the author observed that those $ab_j^{(i)}H_i$ ($0 \leqslant i < n$ and $1 \leqslant j < [H_{i+1} : H_i]$) form a partition of $G \setminus aH_0$. Thus, the cosets, together with aH_0 and $m-1$ copies of G, form an exact m-cover of G with the number k of cosets being $m + \sum_{i=0}^{n-1}([H_{i+1} : H_i] - 1)$ and the intersection of the k subgroups being H. If H_i is maximal normal in H_{i+1} for each $i = 0, \ldots, n-1$ then $k = m + d(G, H)$. In general, as $[H_{i+1} : H_i] - 1 \geqslant f([H_{i+1} : H_i])$ we have

$$k - m \geqslant \sum_{i=0}^{n-1} f([H_{i+1} : H_i]) = f\left(\prod_{i=0}^{n-1}[H_{i+1} : H_i]\right) = f([G : H]).$$

In this paper we study lower bounds for the number k of cosets in an exact m-cover (1.1) of a group G, with the intersection $\bigcap_{i=1}^{k} G_i$ or a subgroup G_i given. In the next section we prove a key property of exact m-covers. In Section 3 we characterize those subnormal subgroups H of group G for which $[G : H] < \infty$ and $d(G, H) = f([G : H])$. In Section 4 we present our main results with an application in group theory.

Now we state our central results. (Actually we will prove more.)

(I) Let G be a group and (1.1) an exact m-cover of G with all the G_i subnormal in G. Then $k \geqslant m + d(G, \bigcap_{i=1}^{k} G_i)$. Moreover, for any subgroup K of G not contained in all the G_i we have

$$|\{1 \leqslant i \leqslant k : K \not\subseteq G_i\}| \geqslant 1 + d\left(K, K \cap \bigcap_{i=1}^{k} G_i\right).$$

(II) Let (1.1) be an exact m-cover of a group G. Whenever $G/(G_i)_G$ is solvable, we have $k \geqslant m + f([G : G_i])$ and hence $[G : G_i] \leqslant 2^{k-m}$.

Concerning result (II) we have a further conjecture.

CONJECTURE . *Let* (1.1) *be an exact m-cover of a group G with all the $G/(G_i)_G$ solvable. Then $k \geqslant m + f(N)$ where N is the least common multiple of* $[G : G_1], \ldots, [G : G_k]$.

For a cover (1.1) of group G, if it does not form an exact m-cover for any $m = 1, 2, 3, \ldots$, then we do not have a similar inequality in general. When G is cyclic, or $|G|$ is square free and all the G_i are subnormal in G, if $m(\mathcal{A}') < m(\mathcal{A})$ for any proper subsystem $\mathcal{A}'$ of $\mathcal{A}$ then we can show that $k \geqslant m(\mathcal{A}) + f([G : \bigcap_{i=1}^k G_i])$. Since this result and its extension are quite involved, we do not present a proof here.

2. A VITAL PROPERTY OF EXACT m-COVERS

Of course, whether two left cosets of subgroups are disjoint, is very essential. Here we give the following lemma.

LEMMA 2.1. *Let H and K be subgroups of a group G. Then:*

(i) *$HK = G$ if and only if $xH \cap yK \neq \emptyset$ for all $x, y \in G$.*
(ii) *$HK = KH$ coincides with G or H in the following cases:*
  (a) *H is maximal and normal in G;*
  (b) *H is maximal in G and K is normal in G;*
  (c) *H is maximal normal and K is subnormal in G.*

PROOF. (i) If $HK = G$, then for any $x, y \in G$ there exist $h \in H$ and $k \in K$ such that $hk = x^{-1}y$ and hence

$$xH \cap yK = xhH \cap xhkK = xh(H \cap K) \neq \emptyset.$$

Conversely, if $xH \cap yK \neq \emptyset$ for all $x, y \in G$, then for any $g \in G$ there is a $h \in H$ for which $h \in gK$ and hence $gK = hK$, therefore $G = HK$.

(ii) In each case H or K is normal in G and thus $HK{=}KH$ is a subgroup of G containing H. Due to the maximality of H, HK coincides with G or H in the cases (a) and (b). In case (c), HK is subnormal in G (by 7.19 of [17]), if $HK \neq G$ then $(HK)^G \neq G$ and so $(HK)^G = H$ (i.e., $HK = H$) by the maximal normality of H. □

REMARK 2.1. For previous combined use of parts (i) and (ii), the reader may consult the proofs of Lemma 6 of [6], Theorem 1 of [13], Lemma 2.II of [1] and Theorem 4 of [18]. For a proper subgroup H of group G, case (a) is equivalent to the following: H is a (maximal) normal subgroup of prime index in G. (Notice that H is maximal in G if it has prime index in G, and that in case (a) G/H is a cyclic group of prime order since G/H and H/H are the only subgroups of G/H.)

LEMMA 2.2. *Let G be a group and $G_1, \ldots, G_k$ be subgroups of G with finite index. Then*

$$\sum_{i=1}^{k} \frac{1}{[G : G_i]} = \frac{1}{[G : \bigcap_{j=1}^k G_j]} \sum_{C \in G/\bigcap_{j=1}^k G_j} |\{1 \leqslant i \leqslant k : C \subseteq a_i G_i\}| \geqslant m(\mathcal{A}). \tag{2.1}$$

Therefore, (1.1) *forms an exact m-cover of G if and only if*

$$\sum_{i=1}^{k} \frac{1}{[G : G_i]} = m \tag{2.2}$$

and $w_{\mathcal{A}}(x) \leqslant m$ (or $w_{\mathcal{A}}(x) \geqslant m$) for all $x \in G$.

PROOF. $H = \bigcap_{j=1}^{k} G_j$ is of finite index in G. Clearly

$$[G:H]\sum_{i=1}^{k}\frac{1}{[G:G_i]} = \sum_{i=1}^{k}[G_i:H] = \sum_{i=1}^{k}\sum_{\substack{C\in G/H\\ C\subseteq a_iG_i}} = \sum_{C\in G/H}\sum_{\substack{i=1\\ a_iG_i\supseteq C}}^{k}.$$

So

$$\begin{aligned}\sum_{i=1}^{k}\frac{1}{[G:G_i]} &= \frac{1}{[G:H]}\sum_{C\in G/H}|\{1\leqslant i\leqslant k:\ C\subseteq a_iG_i\}|\\ &\geqslant \frac{1}{[G:H]}\sum_{C\in G/H} m(\mathcal{A}) = m(\mathcal{A}).\end{aligned}$$

If (1.1) forms an exact m-cover of G, then $|\{1\leqslant i\leqslant k : C\subseteq a_iG_i\}| = m$ for each $C\in G/H$ and hence (2.2) holds. When $w_{\mathcal{A}}(x)\leqslant m$ (or $w_{\mathcal{A}}(x)\geqslant m$) for all $x\in G$, if (2.2) is valid then $|\{1\leqslant i\leqslant k : C\subseteq a_iG_i\}|$ must coincide with m for every $C\in G/H$ and hence $w_{\mathcal{A}}(x)=m$ for any $x\in G$. This completes the proof. □

REMARK 2.2. It has been known that if (1.3) is an exact m-cover of $\mathbb{Z}$ then $\sum_{i=1}^{k}1/[\mathbb{Z}:n_i\mathbb{Z}]$ equals m (see [14]). In 1977 Korec and Znám [8] proved that $\sum_{i=1}^{k}1/[G:G_i]=1$ for any disjoint cover (1.1) of group G.

THEOREM 2.1. *Assume that* (1.1) *forms an exact m-cover of a group G* (*by left cosets of subgroups $G_1,\dots,G_k$*). *For a subgroup H of G we have*

$$\{C\in G/H : C\supseteq a_iG_i \text{ for some } i=1,\dots,k\} = \emptyset \quad\text{or}\quad G/H. \tag{2.3}$$

in the following cases:

(a) *H is the group G or a normal subgroup of prime index in G;*
(b) *$G_1,\dots,G_k$ are normal in G and H is maximal in G;*
(c) *$G_1,\dots,G_k$ are subnormal and H is maximal normal in G.*

PROOF. Suppose, in contrast, that (2.3) is false. Then there exist $a,b\in G$ such that $a_jG_j\subseteq aH$ for some $1\leqslant j\leqslant k$ and that $a_iG_i\not\subseteq bH$ for all $i=1,\dots,k$. Let $I=\{1\leqslant i\leqslant k : G_i\not\subseteq H\}$. Clearly, $j\notin I$ since $a_jH = aH\supseteq a_jG_j$. When $i\in I$, by Lemma 2.1 and Remark 2.1 we have $G_iH=G$ and $a_iG_i\cap bH\neq\emptyset$ in all the three cases. If $s\in\{1,\dots,k\}\setminus I$, then $a_sG_s\subseteq a_sH\neq bH$ and hence $a_sG_s\cap bH=\emptyset$. As (1.1) forms an exact m-cover of G, I must be non-empty and $\{b^{-1}a_iG_i\cap H\}_{i\in I}$ must be an exact m-cover of H by left cosets of subgroups $G_i\cap H$ $(i\in I)$ in H. In view of Lemma 2.2,

$$\sum_{i\in I}\frac{1}{[H:G_i\cap H]} = m = \sum_{i=1}^{k}\frac{1}{[G:G_i]}.$$

However, for each $i\in I$ we have $[H:G_i\cap H]=[G:G_i]$ because $G=G_iH$ contains exactly $[H:G_i\cap H]$ right cosets of G_i by Chapter 1 of [22]. So I must coincide with $\{1,\dots,k\}$, which contradicts the fact that $j\notin I$. The proof is complete. □

REMARK 2.3. When $m=1$, Theorem 2.1 in case (c) was obtained by the author [18] in a particular way.

3. When $d(G, H) = f([G : H])$?

For any group G we let $\mathcal{S}(G)$ denote the class of subnormal subgroups H of G for which $[G : H] < \infty$ and $d(G, H) = f([G : H])$. In this section we aim to characterize those $H \in \mathcal{S}(G)$.

LEMMA 3.1. *Let G be a group.*

(i) *For subgroups H and K of G with $[G : H]$ finite and H or K subnormal in G, we have $[K : H \cap K] \mid [G : H]$.*
(ii) *If $G_1, \ldots, G_k$ are subnormal subgroups of G with finite index, then*

$$\left[G : \bigcap_{i=1}^{k} G_i\right] \Bigg| \prod_{i=1}^{k} [G : G_i]. \tag{3.1}$$

PROOF. (i) When H is subnormal in G, there exists a finite chain

$$H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{n-1} \subseteq H_n = G$$

of subgroups of G such that H_{i-1} is normal in H_i for all $i = 1, \ldots, n$. By the second isomorphism theorem (for group H_i) (cf. 3.40 of [17])

$$[H_i \cap K : (H_i \cap K) \cap H_{i-1}] = |(H_i \cap K)H_{i-1}/H_{i-1}|$$

and hence $[H_i \cap K : H_{i-1} \cap K] \mid [H_i : H_{i-1}]$. We therefore have

$$\prod_{i=1}^{n} [H_i \cap K : H_{i-1} \cap K] \Bigg| \prod_{i=1}^{n} [H_i : H_{i-1}], \text{ i.e., } \ [K : H \cap K] \mid [G : H].$$

Now we assume that K is subnormal in G. Let $K_0 = K \subseteq K_1 \subseteq \cdots \subseteq K_n = G$ be a chain of subgroups such that K_{i-1} is normal in K_i for any $i = 1, \ldots, n$. In view of Chapter 1 of [22], the subgroup $K_{i-1}(H \cap K_i)$ of K_i contains exactly $[K_{i-1} : K_{i-1} \cap (H \cap K_i)] = [K_{i-1} : H \cap K_{i-1}]$ left cosets of $H \cap K_i$ and so $[K_{i-1} : H \cap K_{i-1}] \mid [K_i : H \cap K_i]$. It follows that $[K : H \cap K] = [K_0 : H \cap K_0]$ divides $[K_n : H \cap K_n] = [G : H]$.

This proves the first part.

(ii) By part (i), if H and K are subnormal subgroups of G with finite index, then $[G : H \cap K] = [G : K][K : H \cap K]$ divides $[G : H][G : K]$. Thus we can easily show part (ii) by induction. This completes the proof. □

REMARK 3.1. Let G be a group and H, K be subgroups of G with $[G : H] < \infty$. Although $[K : H \cap K] \leqslant [G : H]$, in general we may have $[K : H \cap K] \nmid [G : H]$ even if G is solvable. For example, the symmetric group $G = S_3$ on $\{1, 2, 3\}$ has subgroups $H = \{e, (12)\}$ and $K = \{e, (13)\}$ with $H \cap K = \{e\}$, apparently $[K : H \cap K] = |K| = 2$, $[G : H] = 3!/2 = 3$ and $[G : H \cap K] \nmid [G : H][G : K]$.

Recall that a group G is called *perfect* if G coincides with its commutator subgroup $G' = [G, G]$, and that every finite group has a unique solvable residual (see 7.50 of [17]).

THEOREM 3.1. *Let G be a group and H be a subgroup of G.*

(i) *$H \in \mathcal{S}(G)$ if and only if there is a composition series from H to G whose factors are of prime orders.*
(ii) *When $H \in \mathcal{S}(G)$, we have $H \cap K \in \mathcal{S}(K)$ for any subgroup K of G, also $H \cap K, \langle H, K\rangle \in \mathcal{S}(G)$ if $K \in \mathcal{S}(G)$.*

(iii) *If H lies in $\mathcal{S}(G)$ then so do H_G and H^G.*
(iv) *$H \in \mathcal{S}(G)$ if and only if H is subnormal in G and G/H_G is finite and solvable.*
(v) *If $H \in \mathcal{S}(G)$ then H contains all perfect subgroups of G. When G is finite and H is subnormal in G, we have $H \in \mathcal{S}(G)$ if H contains the (perfect) normal subgroup K of G for which G/K is the solvable residual of G.*
(vi) *Assume that H is subnormal in G and $[G : H]$ is finite. Then $H \in \mathcal{S}(G)$ if G is locally solvable, or $[G : H]$ is square free or odd or divisible by at most two distinct primes.*

PROOF. (i) In the case $H = G$, clearly $H \in \mathcal{S}(G)$ since $d(G, H) = 0 = f([G : H])$, and the composition series from H to G has length zero and no factor.

Now let H be proper in G. Apparently H is subnormal and of finite index in G if and only if there is a composition series from H to G whose factors are finite. Suppose that H is such a subgroup of G and that $H = H_0 \subset H_1 \subset \cdots \subset H_n = G$ is a composition series from H to G. Then (1.8) holds and

$$f([G : H]) = f\left(\prod_{i=1}^{n} |H_i/H_{i-1}|\right) = \sum_{i=1}^{n} f(|H_i/H_{i-1}|).$$

Since $m - 1 \geqslant f(m)$ for all $m \in \mathbb{Z}^+$, and

$$f(m_1m_2) = f(m_1) + f(m_2) \leqslant m_1 - 1 + m_2 - 1 < m_1m_2 - 1$$

for any integers $m_1, m_2 > 1$, we therefore have

$$\begin{aligned} d(G, H) = f([G : H]) &\iff |H_i/H_{i-1}| - 1 = f(|H_i/H_{i-1}|) \text{ for all } i = 1, \dots, n \\ &\iff |H_i/H_{i-1}| \text{ is a prime number for any } i = 1, \dots, n. \end{aligned}$$

This proves part (i).

(ii) When $H = G$, part (ii) is obvious. Assume that $H \in \mathcal{S}(G)$ but $H \neq G$. By part (i) there exists a composition series $H = H_0 \subset H_1 \subset \cdots \subset H_n = G$ from H to G such that $|H_1/H_0|, \dots, |H_n/H_{n-1}|$ are primes. Let K be a subgroup of G. For each $i = 1, \dots, n$, evidently $H_{i-1} \cap K = H_{i-1} \cap (H_i \cap K)$ is normal in $H_i \cap K$ since H_{i-1} is normal in H_i; by Lemma 3.1(i) $[H_i \cap K : H_{i-1} \cap K]$ divides $[H_i : H_{i-1}]$ and hence coincides with one or the prime $|H_i/H_{i-1}|$. Thus, in view of part (i), $H \cap K = H_0 \cap K \in \mathcal{S}(H_n \cap K) = \mathcal{S}(K)$.

Now suppose that $K \in \mathcal{S}(G)$. Then $H \cap K$ is subnormal and of finite index in G. As $H \cap K \in \mathcal{S}(K)$ and $K \in \mathcal{S}(G)$, we have $H \cap K \in \mathcal{S}(G)$ by part (i). Put $L = (H \cap K)_G$. Then H/L, K/L and their join $\langle H/L, K/L\rangle = \langle H, K\rangle/L$ are subnormal in the finite group G/L (cf. 1.8 and 7.22 of [17]), so $\langle H, K\rangle$ is subnormal in G. Since $(H \cap K)/L \in \mathcal{S}(K/L)$, by part (i) and 7.25 of [17] we have $H/L \in \mathcal{S}(\langle H, K\rangle/L)$ and hence $H \in \mathcal{S}(\langle H, K\rangle)$. As $H \in \mathcal{S}(G)$, $\langle H, K\rangle$ must lie in $\mathcal{S}(G)$.

(iii) Let $H \in \mathcal{S}(G)$. By part (i) $g^{-1}Hg \in \mathcal{S}(G)$ for any $g \in G$. As $[G : H] < \infty$ there are only finitely many distinct conjugates of H. Their intersection is H_G and their join is H^G. Applying part (ii) we obtain that $H_G, H^G \in \mathcal{S}(G)$.

(iv) Let H be subnormal in G. By part (iii), $H_G \in \mathcal{S}(G)$ providing $H \in \mathcal{S}(G)$. By part (i), if $H_G \in \mathcal{S}(G)$ then $H \in \mathcal{S}(G)$ and $H_G \in \mathcal{S}(H)$. So $H \in \mathcal{S}(G)$ if and only if $H_G \in \mathcal{S}(G)$.

Note that G/H_G is a finite solvable group if and only if there is a composition series from H_G/H_G to G/H_G whose factors are of prime orders (cf. 7.56 of [17]). Thus, by part (i), $H_G \in \mathcal{S}(G)$ (i.e., $H \in \mathcal{S}(G)$) if and only if G/H_G is finite and solvable.

(v) As usual, for any group F we let $F^{(0)} = F$ and $F^{(n+1)} = (F^{(n)})'$ for $n = 0, 1, 2, \dots$. Suppose $H \in \mathcal{S}(G)$. Then G/H_G is solvable by part (iv), therefore $(G/H_G)^{(n)} = H_G/H_G$ for some non-negative integer n (see 7.52 of [17]). Notice that $(G/H_G)^{(n)} = (GH_G/H_G)^{(n)} = G^{(n)}H_G/H_G$ (cf. Exercise 164 of [17]). So $G^{(n)} \subseteq H_G \subseteq H$. If M is a perfect subgroup of G, then $M = M^{(n)} \subseteq G^{(n)} \subseteq H$.

For the rest of part (v), let G be finite, and H be a subnormal subgroup of G which contains the smallest normal subgroup K of G such that G/K is solvable. As a characteristic subgroup of K, K' is also normal in G (cf. 3.15 and 3.51 of [17]). Since K/K' is abelian (cf. 3.52 of [17]), G/K' is solvable and so K is perfect by the property of K. Note that $K \subseteq H_G$. Because $G/H_G \cong (G/K)/(H_G/K)$ is solvable, it follows from part (iv) that $H \in \mathcal{S}(G)$.

(vi) As $[G : H] < \infty$, G/H_G is finite and so there are finitely many elements $x_1, \ldots, x_n$ of G such that $\langle x_1^*, \ldots, x_n^* \rangle = G/H_G$ where $x_i^* = x_i H_G$ for $i = 1, \ldots, n$. If G is locally solvable, then finitely generated $K = \langle x_1, \ldots, x_n \rangle$ and the quotient $G/H_G = KH_G/H_G \cong K/(K \cap H_G)$ are solvable, thus $H \in \mathcal{S}(G)$ by part (iv).

Let $[G : H]$ be square free. Then the factors of a composition series from H to G are simple groups with square-free orders. By Corollary 1 to Theorem 2.10 in Ch. 5 of [22], any group $F \neq \{e\}$ of square-free order has a normal subgroup with the index being the least prime divisor of $|F|$. So the factors must have prime orders and hence $H \in \mathcal{S}(G)$ by part (i).

Now suppose that $[G : H]$ is odd or divisible by at most two distinct primes. Clearly so is $|G/H_G|$ because we can view H_G as an intersection of finitely many conjugates of H and $|G/H_G|$ must divide a power of $[G : H]$ by Lemma 3.1(ii). Applying the well-known theorems of Feit–Thompson [3] and Burnside (cf. 8.5.3 of [16]), we then obtain the solvability of G/H_G. So $H \in \mathcal{S}(G)$ by part (iv).

The proof of Theorem 3.1 is now complete. □

4. THE MAIN RESULTS

THEOREM 4.1. *Let G be a group and* (1.1) *an exact m-cover of G by left cosets of subgroups $G_1, \ldots, G_k$. Then, for any fixed $i \in \{1, \ldots, k\}$,*

$$k \geqslant m + f([G : G_i]) \quad \text{if} \quad G_i \text{ contains some } H \in \mathcal{S}(G). \tag{4.1}$$

PROOF. Let $1 \leqslant t \leqslant k$, $H_t \in \mathcal{S}(G)$ and $H_t \subseteq G_t$. We use induction on $[G : H_t]$ to show that $k \geqslant m + f([G : G_t])$.

If $[G : H_t] = 1$, then $G_t = G$ and hence $m + f([G : G_t]) = m = w_{\mathcal{A}}(e) \leqslant k$.

Now assume that $[G : H_t] > 1$. As $H_t \in \mathcal{S}(G)$, by Theorem 3.1(i) there exists a normal subgroup H of G for which H_t is subnormal in H and $p = [G : H]$ is a prime. Write $G/H = \{g_1 H, \ldots, g_p H\}$. Put $I_0 = \{1 \leqslant i \leqslant k : G_i \not\subseteq H\}$ and $I_s = \{1 \leqslant i \leqslant k : a_i G_i \subseteq g_s H\}$ for $s = 1, \ldots, p$. Clearly the union of these pairwise disjoint sets coincides with $\{1, \ldots, k\}$. For any $s = 1, \ldots, p$, by Lemma 2.1 we have $a_i G_i \cap g_s H \neq \emptyset$ if and only if $i \in I_0 \cup I_s$, so $I_0 \cup I_s \neq \emptyset$ and $\{g_s^{-1} a_i G_i \cap H\}_{i \in I_0 \cup I_s}$ forms an exact m-cover of H by left cosets of subgroups $G_i \cap H$ ($i \in I_0 \cup I_s$).

Apparently $H_t \subseteq G_t \cap H$ and $[H : H_t] < [G : H_t]$. Choose $1 \leqslant s \leqslant p$ so that $t \in I_0 \cup I_s$. By the induction hypothesis,

$$|I_0 \cup I_s| \geqslant m + f([H : G_t \cap H]).$$

If $t \in I_0$ then $[G : G_t] = [G_t H : G_t] = [H : G_t \cap H]$ and hence

$$k \geqslant |I_0 \cup I_s| \geqslant m + f([H : G_t \cap H]) = m + f([G : G_t]).$$

In the case $t \in I_s$ (whence $G_t \subseteq H$), by Theorem 2.1 none of $I_1, \ldots, I_p$ is empty, thus

$$\begin{aligned} k &= |I_0| + |I_1| + \cdots + |I_p| \geqslant |I_0 \cup I_s| + p - 1 \\ &\geqslant m + f([H : G_t]) + f([G : H]) = m + f([G : G_t]). \end{aligned}$$

This concludes the induction step. □

REMARK 4.1. Let (1.1) be an exact m-cover of a group G. Then $[G : G_i]$ and $[G : (G_i)_G]$ are finite. When $G/(G_i)_G$ is solvable, we have $H = (G_i)_G \in \mathcal{S}(G)$ by Theorem 3.1(iv), therefore $k \geqslant m + f([G : G_i])$.

COROLLARY 4.1. *Let G be a group and* (1.1) *an exact m-cover of G by left cosets. Suppose that*

$$\emptyset \neq I \subseteq \{1 \leqslant i \leqslant k : G_i \text{ contains a subgroup in } \mathcal{S}(G)\}.$$

Then for any subgroup K of G we have

$$(k-m)|I| \geqslant f\left(\left[K : K \cap \bigcap_{i\in I} G_i\right]\right), \tag{4.2}$$

thus

$$\left[G : \bigcap_{i\in I} G_i\right] \leqslant 2^{(k-m)|I|}. \tag{4.3}$$

PROOF. Let K be an arbitrary subgroup of G and t be any element of I. Put

$$J = \{1 \leqslant j \leqslant k : a_j G_j \cap a_t K \neq \emptyset\}.$$

Then $t \in J$ and $\{a_t^{-1} a_j G_j \cap K\}_{j\in J}$ forms an exact m-cover of K by left cosets of $G_j \cap K$ ($j \in J$). Let H_t be a subgroup in $\mathcal{S}(G)$ contained in G_t. By Theorem 3.1(ii) we have $H_t \cap K \in \mathcal{S}(K)$. In view of Theorem 4.1, $k \geqslant |J| \geqslant m + f([K : G_t \cap K])$.

Write $I = \{i_1, \ldots, i_{|I|}\}$. By the above,

$$\begin{aligned} k-m &\geqslant f([K : K \cap G_{i_1}]),\\ k-m &\geqslant f([K \cap G_{i_1} : K \cap G_{i_1} \cap G_{i_2}]),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ k-m &\geqslant f([K \cap G_{i_1} \cap \cdots \cap G_{i_{|I|-1}},\\ &\qquad K \cap G_{i_1} \cap \cdots \cap G_{i_{|I|-1}} \cap G_{i_{|I|}}]). \end{aligned}$$

Adding these inequalities we then get the desired (4.2).

If we take $K = G$, then (4.2) gives that $(k-m)|I| \geqslant f([G : \bigcap_{i\in I} G_i])$, which implies (4.3) by (1.9). This completes the proof. □

COROLLARY 4.2. *Let $k \geqslant m > 0$ be integers. Then 2^{k-m} is the maximal value that can be the index of a subgroup in a locally solvable group with an exact m-cover by k cosets one of which is a coset of the subgroup.*

PROOF. Suppose that a locally solvable group G possesses an exact m-cover consisting of a coset C_1 of subgroup $G_1, \ldots,$ a coset C_k of subgroup G_k. For $i = 1, \ldots, k$, we let $G_i^* = G_i$ if C_i is a left coset $a_i G_i$, and $G_i^* = a_i^{-1} G_i a_i$ if C_i is a right coset $G_i a_i$ of G_i. As $\{a_i G_i^*\}_{i=1}^k$ forms an exact m-cover of G, each G_i^* has finite index in G and hence $(G_i^*)_G \in \mathcal{S}(G)$ by Theorem 3.1(vi). In the light of Theorem 4.1, for each $n_i = [G : G_i] = [G : G_i^*]$, we have $k \geqslant m + f(n_i)$ and hence $n_i \leqslant 2^{k-m}$ by (1.9).

Now it suffices to notice that the following k residue classes

$$\underbrace{\mathbb{Z}, \ldots, \mathbb{Z}}_{m-1},\ 2^{k-m}\mathbb{Z},\ 2^0 + 2\mathbb{Z}, \ldots, 2^{k-m-1} + 2^{k-m}\mathbb{Z}$$

together form an exact m-cover of the infinite cyclic group $\mathbb{Z}$ with the largest modulus being 2^{k-m}. This follows from Example 1.2 in the case $G = \mathbb{Z}$ and $H = 2^{k-m}\mathbb{Z}$. □

COROLLARY 4.3. *Let* (1.1) *be an exact m-cover of a group G by cosets of subgroups $G_1, \ldots, G_k$. Then $k \geqslant m + f([G : G_i])$ if G_i contains a subnormal subgroup of G with index odd or square free or divisible by at most two distinct primes.*

PROOF. Let H be any subnormal subgroup of G with $[G : H]$ odd or square free or in the form $p^\alpha q^\beta$ where p, q are primes and α, β are non-negative integers. In view of Theorem 3.1(vi) we have $H \in \mathcal{S}(G)$. If $G_i \supseteq H$, then $k \geqslant m + f([G : G_i])$ by Theorem 4.1. This concludes the proof. □

For an exact m-cover of a group G, if all the G_i are subnormal in G then we have a sharp lower bound of k in terms of the intersection $H = \bigcap_{i=1}^k G_i$.

THEOREM 4.2. *Let G be a group and $G_1, \ldots, G_k$ be subnormal subgroups of G such that* (1.1) *forms an exact m-cover of G for some $a_1, \ldots, a_k \in G$. Let K be any subgroup of G and set $I(K) = \{1 \leqslant i \leqslant k : K \not\subseteq G_i\}$.*

(i) *We have*

$$k \geqslant m + d\left(K, K \cap \bigcap_{i=1}^k G_i\right). \tag{4.4}$$

(ii) *If $I(K) \neq \emptyset$, then there exists an $r \in I(K)$ and $x_i \in K \setminus G_i$ for $i \in I(K) \setminus \{r\}$, such that*

$$\left|\left\{x_i\left(K \cap \bigcap_{s=1}^k G_s\right) : i \in I(K) \setminus \{r\}\right\}\right| \geqslant d\left(K, K \cap \bigcap_{s=1}^k G_s\right), \tag{4.5}$$

and hence

$$|\{1 \leqslant i \leqslant k : K \not\subseteq G_i\}| \geqslant 1 + d\left(K, K \cap \bigcap_{s=1}^k G_s\right). \tag{4.6}$$

PROOF. We use induction on the finite index $[G : L]$ where $L = \bigcap_{i=1}^k G_i$.

If $[G : L] = 1$, then $G_1 = \cdots = G_k = G$, so $m + d(K, K \cap L) = m = w_{\mathcal{A}}(e) \leqslant k$ and $I(K) = \emptyset$.

Now let us proceed to the induction step and suppose that $[G : L] > 1$. Choose $1 \leqslant j \leqslant k$ and a maximal normal proper subgroup H of G such that G_j is subnormal in H. Obviously $\{1 \leqslant i \leqslant k : G_i \subseteq H\}$ can be partitioned into $h = |G/H|$ sets

$$J(C) = \{1 \leqslant i \leqslant k :\ a_i G_i \subseteq C\} \ \ (C \in G/H)$$

which are non-empty by Theorem 2.1. Let $I = \{1 \leqslant i \leqslant k : G_i \not\subseteq H\}$. If $i \in I$, then by Lemma 2.1 we have $G_i H = G$ and hence $a_i G_i \cap gH \neq \emptyset$ for all $g \in G$.

Let us take the first step. Set $g_1 = a_j$ and $I_1 = I \cup J(g_1 H)$. Then $j \in I_1$ and system $\mathcal{A}_1 = \{g_1^{-1} a_i G_i \cap H\}_{i \in I_1}$ forms an exact m-cover of $H_0 = H$ by left cosets of subnormal subgroups $G_i \cap H_0$ $(i \in I_1)$ of H_0. Put

$$H_1 = H_0 \cap \bigcap_{i \in I_1} G_i = \bigcap_{i \in I_1} G_i \quad \text{and} \quad M_1 = \{i \in I_1 : K \cap H_0 \not\subseteq G_i \cap H_0\}.$$

Apparently $M_1 \subseteq I(K)$. Observe that

$$\left[H_0 : \bigcap_{i \in I_1} (G_i \cap H_0)\right] = [H_0 : H_1] < [G : L].$$

By the induction hypothesis, we have

$$|I_1| \geqslant m + d\left(K \cap H_0, (K \cap H_0) \cap \bigcap_{i \in I_1}(G_i \cap H_0)\right) = m + d(K \cap H_0, K \cap H_1);$$

if $M_1 \neq \emptyset$ (i.e., $K \cap H_0 \neq K \cap H_1$) then there is an $r_1 \in M_1$ and $x_i \in (K \cap H_0) \setminus (G_i \cap H_0) \subseteq K \setminus G_i$ for $i \in M_1 \setminus \{r_1\}$ such that

$$|\{x_i(K \cap H_1) : i \in M_1 \setminus \{r_1\}\}| \geqslant d(K \cap H_0, K \cap H_1).$$

Suppose that we have found $g_1, \ldots, g_{s-1} \in G$ ($s > 1$) and pairwise disjoint non-empty subsets $I_1, \ldots, I_{s-1}$ of $\{1, \ldots, k\}$ so that for each $1 \leqslant t < s$, either $I_t \subseteq J(g_t H)$ or $t = 1$, and $\mathcal{A}_t = \{g_t^{-1}a_i G_i \cap H_{t-1}\}_{i \in I_t}$ forms an exact m_t-cover of H_{t-1} by left cosets of subnormal subgroups $G_i \cap H_{t-1}$ ($i \in I_t$) of H_{t-1}, where $0 < m_t \leqslant m_1 = m$ and we let

$$H_t = \bigcap_{i \in I_t}(G_i \cap H_{t-1}) = \bigcap_{i \in I_1 \cup \cdots \cup I_t} G_i.$$

In the case $I_s^* = \bigcup_{t=1}^{s-1} I_t \subset \{1, \ldots, k\}$, we proceed step s as follows. Select an element g_s in the union of those $a_i G_i$ with $i \notin I_s^*$. Then m is greater than $l_s = |\{i \in I_s^* : g_s \in a_i G_i\}|$, and $|\{i \in I_s^* : g_s x \in a_i G_i\}| = l_s$ for all $x \in H_{s-1} = \bigcap_{i \in I_s^*} G_i$. Put

$$I_s = \{1 \leqslant i \leqslant k : \ i \notin I_s^* \ \& \ a_i G_i \cap g_s H_{s-1} \neq \emptyset\} \neq \emptyset.$$

For $i \in I_s$ we have $a_i G_i \cap g_s H \neq \emptyset$ and hence $i \in J(g_s H)$ since $I_s \cap I = \emptyset$. Let $m_s = m - l_s$. Then $\mathcal{A}_s = \{g_s^{-1} a_i G_i \cap H_{s-1}\}_{i \in I_s}$ forms an exact m_s-cover of H_{s-1} by left cosets of subnormal subgroups $G_i \cap H_{s-1}$ ($i \in I_s$) of H_{s-1}. Set

$$H_s = H_{s-1} \cap \bigcap_{i \in I_s} G_i = \bigcap_{i \in \bigcup_{t=1}^{s} I_t} G_i \quad \text{and} \quad M_s = \{i \in I_s : K \cap H_{s-1} \not\subseteq G_i \cap H_{s-1}\}.$$

Apparently $M_s \subseteq I(K)$, and $M_s = \emptyset$ if and only if $K \cap H_{s-1} = K \cap H_s$. In the light of the induction hypothesis,

$$|I_s| \geqslant m_s + d\left(K \cap H_{s-1}, K \cap H_{s-1} \cap \bigcap_{i \in I_s}(G_i \cap H_{s-1})\right) \geqslant 1 + d(K \cap H_{s-1}, K \cap H_s);$$

if $M_s \neq \emptyset$ then there is an $r_s \in M_s$ and $x_i \in (K \cap H_{s-1}) \setminus (G_i \cap H_{s-1}) \subseteq K \setminus G_i$ for $i \in M_s \setminus \{r_s\}$ such that

$$|\{x_i(K \cap H_s) : i \in M_s \setminus \{r_s\}\}| \geqslant d(K \cap H_{s-1}, K \cap H_s).$$

Since $h = |G/H| > 1$ we have $I_1 \subset \{1, \ldots, k\}$. As $\{1, \ldots, k\}$ is a finite set the above process will terminate after n steps where $1 < n \leqslant k$. Thus $\bigcup_{s=1}^{n} I_s = \{1, \ldots, k\}$ and $H_n = \bigcap_{i=1}^{k} G_i = L$. Because

$$\bigcup_{C \in G/H} J(C) = \{1 \leqslant i \leqslant k : G_i \subseteq H\} = \bigcup_{s=1}^{n}\{i \in I_s : G_i \subseteq H\} \subseteq \bigcup_{s=1}^{n} J(g_s H),$$

we have $n \geqslant h = |G/H| \geqslant l = [K : H \cap K] \geqslant 1 + d(K, H \cap K)$. Therefore

$$\begin{aligned} k = |I_1| + \sum_{s=2}^{n} |I_s| &\geqslant m + d(K \cap H_0, K \cap H_1) + \sum_{s=2}^{n}(1 + d(K \cap H_{s-1}, K \cap H_s)) \\ &\geqslant m + d(K, H \cap K) + d(K \cap H_0, K \cap H_n) = m + d(K, K \cap L). \end{aligned}$$

Let

$$M = \bigcup_{\substack{1\leqslant s\leqslant n\\ K\cap H_{s-1}\neq K\cap H_s}} (M_s \setminus \{r_s\}).$$

By the above $M \subseteq I(K)$. For $x, y \in K$, if $1 \leqslant s \leqslant n$ and $x(K \cap H_s) \neq y(K \cap H_s)$, then $x(K\cap L) \neq y(K\cap L)$ (otherwise $x^{-1}y \in K\cap L \subseteq K\cap H_s$). If $1 \leqslant t < s \leqslant n$, $x \in K\cap H_{s-1}$ and $y \in (K\cap H_{t-1})\setminus(G_i\cap H_{t-1})$ for some $i \in I_t$, then $x(K\cap L) \subseteq H_{s-1} \subseteq H_t \subseteq G_i\cap H_{t-1}$ and $y(K \cap L) \cap (G_i \cap H_{t-1}) = \emptyset$. Therefore

$$\begin{aligned}
|\{x_i(K\cap L) : i \in M\}| &= \sum_{\substack{1\leqslant s\leqslant n\\ K\cap H_{s-1}\neq K\cap H_s}} |\{x_i(K \cap L) : i \in M_s \setminus \{r_s\}\}| \\
&\geqslant \sum_{\substack{1\leqslant s\leqslant n\\ K\cap H_{s-1}\neq K\cap H_s}} |\{x_i(K \cap H_s) : i \in M_s \setminus \{r_s\}\}| \\
&\geqslant \sum_{\substack{1\leqslant s\leqslant n\\ K\cap H_{s-1}\neq K\cap H_s}} d(K \cap H_{s-1}, K \cap H_s) = \sum_{s=1}^{n} d(K \cap H_{s-1}, K \cap H_s) \\
&= d(K \cap H_0, K \cap H_n) = d(H \cap K, K \cap L).
\end{aligned}$$

If $1 \leqslant s \leqslant n$ and $M_s = \emptyset$, then we let r_s be an element of I_s. Clearly $M' = \{r_s : 1 < s \leqslant l\}$ has cardinality $l-1$, also $M'\cap M = \emptyset$ and $M\cup M' \subseteq I(K)\setminus\{r_1\}$. If $1 < s \leqslant l$, then $G_{r_s} \subseteq H$ since $r_s \in I_s \subseteq J(g_s H)$. Write $K/(H \cap K) = \{H \cap K, b_1(H \cap K), \ldots, b_l(H \cap K)\}$. When $1 < s \leqslant l$, we have $b_s \notin H$ (otherwise $b_s \in H \cap K$, which is impossible) and hence $x_{r_s} = b_s \in K \setminus G_{r_s}$. Thus $M \cup M' \subseteq I(K) \setminus \{r_1\}$. Recall that $x_i \in K \cap H_{s-1} \subseteq H \cap K$ for $i \in M_s \setminus \{r_s\}$. For $x, y \in K$ with $x(H \cap K) \neq y(H \cap K)$, obviously $x(K \cap L) \neq y(K \cap L)$. For $i \in (I(K) \setminus \{r_1\}) \setminus (M \cup M')$ let x_i be any element of $K \setminus G_i$. Then

$$\begin{aligned}
&|\{x_i(K \cap L) : i \in I(K) \setminus \{r_1\}\}| \\
&\quad\geqslant |\{x_i(K \cap L) : i \in M \cup M'\}| = |\{x_i(K \cap L) : i \in M'\}| + |\{x_i(K \cap L) : i \in M\}| \\
&\quad\geqslant l - 1 + d(H \cap K, K \cap L) \geqslant d(K, H \cap K) + d(H \cap K, K \cap L) = d(K, K \cap L)
\end{aligned}$$

and therefore $|I(K)| \geqslant 1 + d(K, K \cap L)$.

The proof by induction is now complete. □

REMARK 4.2. (a) Clearly $I(K) \subseteq \{1 \leqslant i \leqslant k : G_i \neq G\}$, so we cannot substitute m ($\geqslant |\{1 \leqslant i \leqslant k : G_i = G\}|$) for the first term 1 on the right-hand side of (4.6). (b) Let H be a subnormal subgroup of finite index in group G. If (1.1) forms an exact m-cover of G with $G_1, \ldots, G_k$ subnormal in G and $\bigcap_{i=1}^k G_i = H$, then by taking $K = G$ in Theorem 4.2(i) we get the inequality $k \geqslant m+d(G, H)$. On the other hand, by Example 1.2 there indeed exists an exact m-cover (1.1) of G with all the G_i subnormal in G, $\bigcap_{i=1}^k G_i = H$ and $k = m+d(G, H)$.

COROLLARY 4.4. *Let G be a group and H, K be subnormal subgroups of G with finite index. Let $G_1, \ldots, G_k$ be subnormal subgroups of G for which all the G_i contain H but $I(K) = \{1 \leqslant i \leqslant k : K \not\subseteq G_i\} \neq \emptyset$. Suppose that $X = \bigcup_{i=1}^k a_i G_i$ is a union of left cosets of K and $w_{\mathcal{A}}(x) = m$ for all $x \in X$, where $a_1, \ldots, a_k \in G$. Then there are $C_i \in K/(H \cap K)$ for those $i \in I(K)$ such that $C_i \cap G_i \neq \emptyset$ for a unique $i \in I(K)$ and that*

$$|\{C_i : i \in I(K)\}| \geqslant 1 + d\left(K, K \cap \bigcap_{i=1}^{k} G_i\right). \tag{4.7}$$

PROOF. The set $G \setminus X$ can be written as a union of finitely many distinct left cosets $b_1K, \ldots, b_lK$ of K. Let $G_{k+1} = \cdots = G_{k+lm} = K$ and $a_{k+(j-1)m+1} = \cdots = a_{k+jm} = b_j$ for $j = 1, \ldots, l$. Clearly $\mathcal{A}' = \{a_iG_i\}_{i=1}^{k+lm}$ forms an exact m-cover of G by left cosets of subnormal subgroups of G. Observe that $\bigcap_{i=1}^{k+lm} G_i = K \cap L$, where $L = \bigcap_{i=1}^{k} G_i$. Also, $\{1 \leqslant i \leqslant k+lm : K \not\subseteq G_i\} = I(K) \neq \emptyset$. In the light of Theorem 4.2(ii), there is an $r \in I(K)$ and $x_i \in K \setminus G_i$ for $i \in I(K) \setminus \{r\}$ such that $|\{x_i(K \cap L) : i \in I(K) \setminus \{r\}\}| \geqslant d(K, K \cap L)$. Put $C_r = H \cap K$ and $C_i = x_i(H \cap K)$ for $i \in I(K) \setminus \{r\}$. Note that $e \in C_r \cap G_r$. For $i \in I(K) \setminus \{r\}$ we have $C_i \cap G_i = \emptyset$ because $C_i \subseteq x_iG_i$ and $x_i \notin G_i$. If s and t are distinct elements of $I(K) \setminus \{r\}$, then

$$C_s = C_t \Rightarrow x_s^{-1}x_t \in H \cap K \subseteq K \cap L \Rightarrow x_s(K \cap L) = x_t(K \cap L).$$

So

$$\begin{aligned}|\{C_i : i \in I(K)\}| &= 1 + |\{C_i : i \in I(K) \setminus \{r\}\}| \\ &\geqslant 1 + |\{x_i(K \cap L) : i \in I(K) \setminus \{r\}\}| \geqslant 1 + d(K, K \cap L).\end{aligned}$$

This completes the proof. □

REMARK 4.3. When $m = 1$ and $G_1, \ldots, G_k, K$ are normal in G, the inequality $k \geqslant 1 + d(K, K \cap \bigcap_{i=1}^{k} G_i)$ was obtained by Korec [7] in another way. Theorem 9 of [18] is essentially Corollary 4.4 in the case $m = 1$ and $X = G$.

COROLLARY 4.5. *Let m be a positive integer and H a subgroup of a group G with $[G : H]$ finite.*

(i) *If G is locally nilpotent, then $m + f([G : H])$ is the least positive integer k such that there exists an exact m-cover of G by k left cosets of subgroups whose intersection is H.*

(ii) *Providing H is subnormal in G, if G is locally solvable, or $[G : H]$ odd or square free or in the form $p^{\alpha}q^{\beta}$ where p, q are distinct primes and α, β are non-negative integers, then $m + f([G : H])$ is the smallest $k \in \mathbb{Z}^+$ such that there exists an exact m-cover* (1.1) *of G with all the G_i subnormal in G and $\bigcap_{i=1}^{k} G_i$ equal to H.*

PROOF. In view of Remark 4.2(b), it suffices to make the following observations:

(i) When G is locally nilpotent, by Theorem 7 of [18] H is in $\mathcal{S}(G)$ and any subgroup containing H is subnormal in G.

(ii) Suppose that H is subnormal in G, and that G is locally solvable or $[G : H]$ is odd or square free or divisible by at most two distinct primes. Then $d(G, H) = f([G : H])$ by Theorem 3.1(vi). □

COROLLARY 4.6. *Let H be a subnormal subgroup of a group G with $[G : H] < \infty$. Then H is normal in G if and only if*

$$|N_G(H)/H| + d(H, H_G) \geqslant [G : H] \tag{4.8}$$

where $N_G(H)$ denotes the normalizer of H in G.

PROOF. If H is normal in G then (4.8) holds because $N_G(H) = G$ and $H_G = H$.

Below we assume that H is not normal in G. Let $k = [G : H]$, and $\{Ha_1, \ldots, Ha_k\}$ be a partition of G into right cosets of H. Clearly $G_i = a_i^{-1}Ha_i$ is subnormal in G for each

$i = 1, \ldots, k$. As mentioned in Example 1.1 system (1.1) forms a disjoint cover of G with $\bigcap_{i=1}^k G_i = H_G$. Since $H \supset H_G = \bigcap_{i=1}^k G_i$, it follows from Theorem 4.2(ii) that

$$|\{1 \leqslant i \leqslant k : H \not\subseteq G_i\}| \geqslant 1 + d\left(H, H \cap \bigcap_{i=1}^k G_i\right) = 1 + d(H, H_G).$$

For $i = 1, \ldots, k$, as $[G : G_i] = [G : H] < \infty$, $H \subseteq G_i$ if and only if $H = G_i$. We have $[N_G(H) : H] = |\{1 \leqslant i \leqslant k : G_i = H\}|$, because

$$\begin{aligned} N_G(H) = \{g \in G : g^{-1}Hg = H\} &= \bigcup_{i=1}^k \{ha_i : h \in H \ \&\ (ha_i)^{-1}Hha_i = H\} \\ &= \bigcup_{i=1}^k \{ha_i : h \in H \ \&\ a_i^{-1}Ha_i = H\} = \bigcup_{\substack{i=1 \\ a_i^{-1}Ha_i = H}}^k Ha_i. \end{aligned}$$

Therefore

$$\begin{aligned} d(H, H_G) < |\{1 \leqslant i \leqslant k : H \not\subseteq G_i\}| &= |\{1 \leqslant i \leqslant k : G_i \neq H\}| \\ &= k - |\{1 \leqslant i \leqslant k : G_i = H\}| = [G : H] - |N_G(H)/H|. \end{aligned}$$

Consequently (4.8) fails to hold. We are done. □

REMARK 4.4. Let G be a group and H be a subnormal subgroup of G with finite index. By Corollary 3 of [18], $[G : H] \geqslant 1 + d(G, H_G)$ and consequently $|G/H_G| \leqslant 2^{[G:H]-1}$. In view of Corollary 4.6 and (1.9), if H is not normal in G then $|H/H_G| \leqslant 2^{d(H,H_G)} \leqslant 2^{[G:H]-1-|N_G(H)/H|}$. It seems that $|N_G(H)/H| \geqslant d(G, H)$.

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